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Lie symmetries and multiple solutions in $\lambda - \omega$ reaction-diffusion systems

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Abstract. Lie theory of transformation groups is applied to the study of $\lambda-\omega$ reaction-diffusion systems in two-dimensional media. Our study proves that they are invariant with respect to a five-parameter symmetry group. Multiple types of invariant solutions with physical interest are possible, and some of them can be found in the literature applied to particular models.

1. Introduction

Nonlinear reaction-diffusion equations have been widely studied throughout recent years. These equations arise naturally as description models of many evolution problems in the real world, as in chemistry [1], biology [2], ecology [3], etc.

As is well known, complex behaviour is a peculiarity of systems modelled by reaction-diffusion equations, and the Belousov–Zhabotinskii reaction [4–6] provides a classic example.

Reaction-diffusion equations have been investigated for certain boundary and initial conditions and in most cases explicit solutions cannot been found.

This paper deals with the application of Lie group theory to nonlinear reaction-diffusion equations. Although group analysis of differential equations has been applied a great deal in many fields of mathematical physics [7–11], much less has been applied in connection with problems related to reaction-diffusion models. We think that the application of these techniques to systems of reaction-diffusion equations may help to elucidate many types of solutions, especially for models which possess the appropriate symmetries.

We have selected for investigation the denominated $\lambda - \omega$ models, introduced some years ago by Koppell and Howard [12], which have been widely used in prototype studies of reaction-diffusion processes. Their importance lies in the fact that $\lambda - \omega$ systems arise naturally as the dominant part in the asymptotic analysis of many general reaction-diffusion systems [13]. Spiral wave solutions of particular $\lambda - \omega$ systems have been investigated, for example, by Greenberg [14], Hagan [15] and Kuramoto and Koga [16]. Many other solutions are also known and the list of references is extensive.

We show that the $\lambda - \omega$ systems in two-dimensional media are invariant with respect to a five-parameter symmetry group. The invariance properties give rise to multiple types of solutions and to the *reduced equations*, which are essential in the study of bifurcating solutions applied to particular models.

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2. Lie symmetries and $\lambda - \omega$ reaction-diffusion models

The $\lambda - \omega$ reaction-diffusion systems with two reactants are described by systems of partial differential equations (SPDE) of the form

$$\begin{aligned} u_t &= D\nabla^2 u + \lambda(z)u - \omega(z)v \\ v_t &= D\nabla^2 v + \omega(z)u + \lambda(z)v \\ z &= (u^2 + v^2)^{1/2} \end{aligned} \tag{1}$$

where $\lambda(z)$ is a positive function of z for $0 \leqslant z < z_0$ and negative for $z > z_0$, $\omega(z)$ is a positive function of z; $u = u(x, y, t)$ and $v = v(x, y, t)$ represent, for example, concentrations of two chemical reactants which at the same time diffuse through the plane (x, y). D represents the diffusion coefficient, $\lambda(z)u-\omega(z)v$ and $\omega(z)u+\lambda(z)v$ are nonlinear functions that describe the kinetics of the reaction. The spacially homogeneous system, has a limit cycle solution with amplitude z_0 and frequency $w(z_0)$, thus, $\lambda-\omega$ systems have been proposed as models for chemical or biological systems which exhibit oscillating behaviour in homogeneous situations.

We have found, using Lie group theory of transformations [8], that this system is invariant with respect to the five-parameter group which has associated with it the following characteristics:

$$\begin{aligned} Q^u &= a_1 u_x + a_2 u_y + a_3 u_t + a_4\,(x u_y - y u_x) + a_5 v \\ Q^v &= a_1 v_x + a_2 v_y + a_3 v_t + a_4\,(x v_y - y v_x) - a_5 u \end{aligned} \tag{2}$$

where the set $\{a_i\}_{i=1}^5$ represents arbitrary constants. Every set $\{a_i\}_{i=1}^5$ is associated to a one-parameter group of transformations.

Five simple one-parameter groups can be obtained by making $a_i = 1$, $i = 1, \ldots, 5$, and $a_j = 0$ with $j \neq i$. We denote each of theses groups by G_i, and the associated characteristics by Q_i^u and Q_i^v:

$$\begin{array}{lll} G_1: & Q_1^u = u_x & Q_1^v = v_x \\ G_2: & Q_2^u = u_y & Q_2^v = v_y \\ G_3: & Q_3^u = u_t & Q_3^v = v_t \\ G_4: & Q_4^u = x u_y - y u_x & Q_4^v = x v_y - y v_x \\ G_5: & Q_5^u = v & Q_5^v = -u. \end{array} \tag{3}$$

The characteristics associated with G_1, G_2 and G_3 correspond to translations in the coordinates x, y and t, respectively. Those associated with G_4 and G_5 correspond to rotations in the planes (x, y) and (u, v), respectively.

Also, we denote by G_{ij} the one-parameter groups obtained by making $a_i \neq 0$, $a_j \neq 0$ and $a_k = 0$ with $k \neq i, j$.

It is convenient to change the variables (x, y) to polar variables (r, θ), and (u, v) to polar variables (z, ϕ). The characteristics of G_4 and G_5 are

$$\begin{array}{lll} G_4: & Q_4^u = a_4 u_\theta & Q_4^v = a_4 v_\theta \\ G_5: & Q_5^z = 0 & Q_5^\phi = 1. \end{array} \tag{4}$$

In terms of the variables (z, ϕ), system (1) reads

$$\nabla^2 z + z(\lambda(z) - |\nabla\phi|^2) - z_t = 0 \qquad \nabla^2\phi + 2\nabla\phi\frac{\nabla z}{z} + \omega(z) - \phi_t = 0. \tag{5}$$

Let us now consider the general reaction-diffusion systems of the form

$$F \equiv \nabla^2 u + f(u, v) - u_t = 0 \qquad G \equiv \nabla^2 v + g(u, v) - v_t = 0. \tag{6}$$

If these systems are invariant under the groups associated with the characteristics (2), we can demonstrate that they are of the type $\lambda - \omega$.

These systems are invariant with respect to G_1, G_2, G_3 and G_4, because F and G do not depend explicitly on (x, y, t). The condition of invariance with respect to G_5 is

$$V_5(F) = 0 \qquad V_5(G) = 0 \tag{7}$$

when u is a solution of the system (6). We represent by V_5 the prolongation of the Lie operator for G_5:

$$V_5 = -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v} + \sum_I -v_I\frac{\partial}{\partial u_I} + u_I\frac{\partial}{\partial v_I} \tag{8}$$

where I is a multi-index referring to the multiple derivatives of u and v, with $|I| > 0$.

Then

$$-\nabla^2 v - v\frac{\partial f}{\partial u} + u\frac{\partial f}{\partial v} + v_t = 0 \qquad \nabla^2 u - v\frac{\partial g}{\partial u} + u\frac{\partial g}{\partial v} - u_t = 0. \tag{9}$$

That is, substituting u_t and v_t from (6):

$$-v\frac{\partial f}{\partial u} + u\frac{\partial f}{\partial v} = -g \qquad -v\frac{\partial g}{\partial u} + u\frac{\partial g}{\partial v} = f. \tag{10}$$

These equations may be written in the variables (z, ϕ) as

$$\frac{\partial f}{\partial \phi} = -g \qquad \frac{\partial g}{\partial \phi} = f \tag{11}$$

so thus

$$f + \frac{\partial^2 f}{\partial \phi^2} = 0 \qquad g = -\frac{\partial f}{\partial \phi}.$$

Consequently, the functions f and g take the form of the kinetics of the $\lambda - \omega$ systems:

$$\begin{aligned} f &= \lambda(z)z\cos(\phi) - \omega(z)z\sin(\phi) = \lambda(z)u - \omega(z)v \\ g &= \omega(z)z\cos(\phi) + \lambda(z)z\sin(\phi) = \omega(z)u + \lambda(z)v. \end{aligned} \tag{12}$$

Thus, we have proved that $\lambda - \omega$ systems are characterized, among reaction-diffusion systems, by their symmetry properties. In the next section we show that the study of solutions invariant with respect to some subgroups of the full symmetry group may be useful to describe pattern formation.

3. Invariant solutions

Invariant solutions, u and v, for a subgroup of the full symmetry group, i.e. partially invariant solutions [17], must satisfy system (1) and the characteristic equations

$$Q^u(u, v) = 0 \qquad Q^v(u, v) = 0 \tag{13}$$

for some set of the constants $\{a_i\}$. This requirement imposes special forms to the solutions. Substitution in system (1) gives rise to the reduced equations, which are PDEs with the number of independent variables reduced to one.

There exist two types of invariant solutions according to the value of the constant a_5.

(i) If $a_5 = 0$, it is possible to change the variables (x, y, t) to new variables (ξ_1, ξ_2, η), such that the characteristic equations are

$$u_\eta = 0 \qquad v_\eta = 0. \tag{14}$$

Hence, invariant solutions depend only on (ξ_1, ξ_2). Substitution in (1) leads to a new system of PDEs with two independent variables.

(ii) If $a_5 \neq 0$, it is possible to change variables from (x, y, t) to (ξ_1, ξ_2, η) such that the characteristic equations are

$$\alpha u_\eta + v = 0 \qquad \alpha v_\eta - u = 0. \tag{15}$$

Then

$$u = z(\xi_1, \xi_2) \cos\left(\frac{\eta}{\alpha} + \beta(\xi_1, \xi_2)\right) \qquad v = z(\xi_1, \xi_2) \sin\left(\frac{\eta}{\alpha} + \beta(\xi_1, \xi_2)\right). \tag{16}$$

That is, invariant solutions are periodic functions with respect to η. Substitution of $z = z(\xi_1, \xi_2)$ and $\phi = (\eta/\alpha) + \beta(\xi_1, \xi_2)$ in (5) leads to the reduced equations for z and β.

If a solution is invariant with respect to a two-parameter group, the reduced equations are ordinary differential equations (ODEs).

In the following a solution invariant with respect to a group G_I will be called a G_I solution. If it is invariant with respect to two groups G_I and G_J, it will be called a $G_I + G_J$ solution.

4. Multiple solutions

In this section we consider solutions invariant with respect to different subgroups.

4.1. Homogeneous solutions

These are $G_1 + G_2$ solutions. The reduced equations are

$$\lambda(z) - z_t = 0 \qquad \phi_t - \omega(z) = 0. \tag{17}$$

As $\lambda(z)$ has a zero with negative derivative in z_0, then there exists a stable limit cycle defined by the equations

$$u = z_0 \cos(\omega(z_0)t + \phi_0) \qquad v = z_0 \sin(\omega(z_0)t + \phi_0). \tag{18}$$

4.2. Travelling waves

These are G_{15} solutions, and the characteristic equations take the form

$$z_x = 0 \qquad a_1\phi_x - 1 = 0. \tag{19}$$

Then

$$u = z(y, t) \cos\left(\frac{x}{a_1} + \beta(y, t)\right) \qquad v = z(y, t) \sin\left(\frac{x}{a_1} + \beta(y, t)\right). \tag{20}$$

If in addition they are G_{35} solutions,

$$u = z(y) \cos\left(\frac{x}{a_1} + \frac{t}{a_3} + \alpha(y)\right) \qquad v = z(y) \sin\left(\frac{x}{a_1} + \frac{t}{a_3} + \alpha(y)\right) \tag{21}$$

which are travelling wavetrain solutions.

The reduced equations are

$$z_{yy} + z\left(\lambda(z) - \frac{1}{a_1^2} - \alpha_y^2\right) = 0 \qquad \alpha_{yy} + 2\alpha_y \frac{z_y}{z} + \left(\omega(z) - \frac{1}{a_3}\right) = 0. \tag{22}$$

4.3. Stationary bands

The characteristic equations for the two-parameter group $G_{15} + G_3$ are

$$z_x = 0 \qquad z_t = 0$$
$$\phi_x = \frac{1}{a_1} \qquad \phi_t = 0. \tag{23}$$

The solutions take the form

$$u = z(y)\cos\left(\frac{x}{a_1} + \beta(y)\right) \qquad v = z(y)\sin\left(\frac{x}{a_1} + \beta(y)\right). \tag{24}$$

The reduced equations are

$$z_{yy} + z\left(\lambda(z) - \frac{1}{a_1^2} - \beta_y^2\right) = 0 \qquad \beta_{yy} + 2\beta_y\frac{z_y}{z} + \omega(z) = 0. \tag{25}$$

4.4. Wave packets

The characteristic equations for G_{135} solutions are

$$a_1 z_x + a_3 z_t = 0 \qquad a_3\phi_x + a_1\phi_t - 1 = 0. \tag{26}$$

A change of variables to $x' = x - c_g t$ and $t' = t$, where $c_g = a_1/a_3$, leads to new characteristic equations:

$$a_1 z_{t'} = 0 \qquad a_3\phi_{t'} - 1 = 0. \tag{27}$$

The amplitude and phase in the new variables take the form

$$z = z(x', y) \qquad \phi = \Omega t' + \beta(x', y) \tag{28}$$

where $\Omega = 1/a_3$. The invariant solutions are

$$u = z(x', y)\cos(\Omega t' + \beta(x', y)) \qquad v = z(x', y)\sin(\Omega t' + \beta(x', y)). \tag{29}$$

We define the complex function

$$\bar{u} = u + \mathrm{i}v = z(x', y)\,\mathrm{e}^{\mathrm{i}(\beta(x',y)+\Omega t')}. \tag{30}$$

It is easy to compare this expression with the wave packet travelling in the x-direction:

$$\bar{u}' = \int G(k, y)\,\mathrm{e}^{\mathrm{i}(kx-\omega(k)t)}\,\mathrm{d}k. \tag{31}$$

The group speed $c_g = \mathrm{d}w/\mathrm{d}k$ is assumed to be approximately constant in the interval where G is significantly different from zero. Then $w(k) = w_0 + c_g k'$, with $k' = k - k_0$ for some arbitrary wavenumber k_0 in that interval, so

$$\bar{u}' = \int G'(k', y)\,\mathrm{e}^{\mathrm{i}(k_0x+k'x-w_0t-c_gk't)}\,\mathrm{d}k' = A(x', y)\,\mathrm{e}^{\mathrm{i}\alpha(x',y)}\,\mathrm{e}^{\mathrm{i}(k_0x-w_0t)} \tag{32}$$

with $G'(k', y) = G(k_0 + k', y)$, and

$$A(x', y)\,\mathrm{e}^{\mathrm{i}\alpha(x',y)} = \left(\int G'(k', y)\,\mathrm{e}^{\mathrm{i}(k'(x-c_gt))}\,\mathrm{d}k'\right). \tag{33}$$

This expression may be identified with the G_{135} solutions if

$$\omega_0 = k_0c_g - \Omega \qquad \alpha(x', y) = \beta(x', y) - k_0x'. \tag{34}$$

The reduced equations are

$$
\begin{aligned}
&z_{x'x'} + z_{yy} + z(\lambda(z) - \beta_{x'}{}^2 - \beta_y{}^2) + c_g z_{x'} = 0 \\
&\beta_{x'x'} + \beta_{yy} + 2\beta_{x'}\frac{z_{x'}}{z} + 2\beta_y\frac{z_y}{z} + \omega(z) - \Omega + c_g\beta_{x'} = 0.
\end{aligned}
\tag{35}
$$

If the solutions are $G_2 + G_{135}$ invariant, the functions z and β do not depend on y, that is the wavefronts are straight lines.

4.5. Solutions with rotational symmetry

These are G_4 solutions, with characteristic equations

$$u_\theta = 0 \qquad v_\theta = 0. \tag{36}$$

4.5.1. Stationary target patterns. These are G_4+G_3 solutions. The additional characteristic equations are

$$u_t = 0 \qquad v_t = 0. \tag{37}$$

The solutions are of the form $u = u(r)$, $v = v(r)$. The reduced equations are

$$u_{rr} + \frac{u_r}{r} + u\lambda(z) - \omega(z)v = 0 \qquad v_{rr} + \frac{u_r}{r} + u\omega(z) + \lambda(z)v = 0. \tag{38}$$

4.5.2. Travelling circular waves. These are $G_4 + G_{35}$ solutions, with characteristic equations

$$z_t = 0 \qquad a_3\phi_t - 1 = 0. \tag{39}$$

The solutions are of the form

$$u = z(r)\cos\left(\frac{t}{a_3} + \beta(r)\right) \qquad v = z(r)\sin\left(\frac{t}{a_3} + \beta(r)\right). \tag{40}$$

The reduced equations are

$$z_{rr} + \frac{z_r}{r} + z(\lambda(z) - \beta_r^2(z)) = 0 \qquad \beta_{rr} + \frac{\beta_r}{r} + 2\beta_r\frac{z_r}{z} + \omega(z) - \frac{1}{a_3} = 0. \tag{41}$$

If $\beta(r)$ is not constant these solutions are travelling circular waves with speed $c = -1/a_3\beta_r$.

4.5.3. Stationary circular waves. If $\beta(r)$ is a constant β, then $\omega(z)$ must be also constant with value $1/a_3$, and the solutions are of the form

$$u = z(r)\cos\left(\frac{t}{a_3} + \beta\right) \qquad v = z(r)\sin\left(\frac{t}{a_3} + \beta\right) \tag{42}$$

which are stationary circular waves.

The reduced equation is

$$z_{rr} + \frac{z_r}{r} + z\lambda(z) = 0. \tag{43}$$

4.6. Rotating waves

These are G_{34} solutions, with characteristic equations

$$a_4 u_\theta + a_3 u_t = 0 \qquad a_4 v_\theta + a_3 v_t = 0 \tag{44}$$

which may be written in the variables $\theta' = \theta - \Omega t$ and $t' = t$, where $\Omega = a_4/a_3$, as

$$a_3 u_{t'} = 0 \qquad a_3 v_{t'} = 0. \tag{45}$$

Then, $u = u(r, \theta')$ and $v = v(r, \theta')$. The reduced equations are

$$\nabla' u + u\lambda(z) - \omega(z) v + u_{\theta'}\Omega = 0 \qquad \nabla' v + u\omega(z) + \lambda(z) v + v_{\theta'}\Omega = 0 \tag{46}$$

where ∇' is the nabla operator in the new variables.

4.7. Solutions with S_n symmetry

These are G_{45} solutions. The characteristic equations are

$$a_4 z_\theta = 0 \qquad a_4 \phi_\theta - 1 = 0. \tag{47}$$

Then, $z = z(r, t)$ and $\phi = (\theta/a_4) + \beta(r, t)$. The solutions are of the form

$$u = z(r,t)\cos\left(\frac{\theta}{a_4} + \beta(r,t)\right) \qquad v = z(r,t)\sin\left(\frac{\theta}{a_4} + \beta(r,t)\right). \tag{48}$$

These solutions must be continuous in the plane (x, y), that is $u(r, \theta, t) = u(r, \theta + 2\pi, t)$ and $v(r, \theta, t) = v(r, \theta + 2\pi, t)$, so then $a_4 = 1/n$, where n is an integer. The solutions are of the form

$$u = z(r,t)\cos(n\theta + \beta(r,t)) \qquad v = z(r,t)\sin(n\theta + \beta(r,t)). \tag{49}$$

There are n equations for the curves of constant phase 2π:

$$\theta = -\frac{1}{n}\beta(r,t) + 2\pi\frac{m}{n} \qquad m = 0, 1, 2, \ldots, n-1. \tag{50}$$

The reduced equations are

$$z_{rr} + \frac{z_r}{r} + z\left(\lambda(z) - {\beta_r}^2 - \frac{n^2}{r^2}\right) - z_t = 0 \qquad \beta_{rr} + \frac{\beta_r}{r} + 2\beta_r\frac{z_r}{z} + \omega(z) - \beta_t = 0. \tag{51}$$

4.7.1. Stationary solutions with S_n symmetry.

The solutions and reduced equations with $G_{45} + G_3$ symmetry have the same form as above, with the condition that β and z are t independent.

4.7.2. Multi-armed rotating spiral waves.

These are $G_{45}+G_{35}$ solutions. The characteristic equations for G_{35} are

$$z_t = 0 \qquad a_3\phi_t - 1 = 0.$$

Then the solutions are of the form

$$u = z(r)\cos(n\theta + \Omega t + \beta(r)) \qquad v = z(r)\sin(n\theta + \Omega t + \beta(r)) \tag{52}$$

with $\Omega = 1/a_3$. The reduced equations are

$$z_{rr} + \frac{z_r}{r} + z\left(\lambda(z) - {\beta_r}^2 - \frac{n^2}{r^2}\right) = 0 \qquad \beta_{rr} + \frac{\beta_r}{r} + 2\beta_r\frac{z_r}{z} + \omega(z) - \Omega = 0. \tag{53}$$

The phase curves rotate rigidly with angular speed Ω.

5. Conclusions

The $\lambda-\omega$ systems are characterized, among reaction-diffusion systems, by symmetry group. Solutions invariant with respect to different subgroups of the full symmetry group exhibit many different patterns with physical interest. These solutions have a lower degree of symmetry than the system, so they are probably the emerging solutions in spontaneous symmetry-breaking processes. The study of the reduced equations with appropiate boundary conditions applied to specific models is necessary to delimitate the ranges of the parameter values, inherent to each model, associated with different types of solutions. We are now concluding a study relative to a model for the Belousov–Zhabotinskii reaction [18], which is a $\lambda-\omega$ system.

Appendix. Determination of Lie symmetries

In this appendix we briefly sketch, without technical detail, the method used for obtaining the characteristics of $\lambda-\omega$ systems. A complete reference can be found in [8].

A.1. Group of transformations

Let G be a local Lie Group, $x = (x_1, x_2, \ldots, x_n)$ the set of independent variables, and $u = (u^1, u^2, \ldots, u^m)$ the set of dependent variables, in a space of functions $u = u(x)$. A local Lie group of transformations in the space (x, u) is given by the set of equations:

$$x^\epsilon = X(x, u, \epsilon) \qquad u^\epsilon = U(x, u, \epsilon) \tag{A1}$$

where ϵ is a continuous parameter of a local group, $\epsilon = 0$ being the value of the parameter for the identity element. The expression local means that the group properties are valid at least in some neighbourhood of $\epsilon = 0$. If the functions X and U depend not only on x and u but also on some derivatives, the transformations (A1) have no geometrical interpretation, and must be seen as transformations in a space of functions $u(x)$. In this case they are called generalized transformations.

A.2. Infinitesimals

For every transformation (A1) there is an infinitesimal transformation given by

$$\delta x = \xi(x, u)\epsilon \qquad \delta u = \eta(x, u)\epsilon \tag{A2}$$

with ϵ small enough; $\xi = (\xi^1, \xi^2, \ldots, \xi^n)$ and $\eta = (\eta^1, \eta^2, \ldots, \eta^m)$ are called the infinitesimals of the transformation and are given by

$$\xi = \left(\frac{\partial X}{\partial \epsilon}\right)_{\epsilon=0} \qquad \eta = \left(\frac{\partial U}{\partial \epsilon}\right)_{\epsilon=0}. \tag{A3}$$

A.3. Characteristics

The characteristic of the transformation group is defined as $Q = \eta - \xi^i u_i$. An equivalent transformation [8] to (A1) that leaves invariant the x variables is given infinitesimally by

$$\delta u = Q(x, u, \{u_i\})\epsilon \qquad \text{where } Q = \left(\frac{\partial U}{\partial \epsilon}\right)_{\epsilon=0}. \tag{A4}$$

This is a generalized transformation which has an equivalent geometrical transformation. The expression $\{u_i\}$ represents the set of derivatives $\partial u^\alpha/\partial x^i$ with $\alpha = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n$.

We represent by $\{u_I\}$, where $I = (i_1, i_2, \ldots, i_n)$ is a multi-index, the set of derivatives, given explicitly by the expressions

$$\{u_I\} \rightarrow \frac{\partial^{|I|} u^\alpha}{\partial x_1^{i_1} \partial x_2^{i_2} \ldots \partial x_n^{i_n}} \qquad \alpha = 1, 2, \ldots, m; |I| = \sum_{j=1}^{n} i_j > 0.$$

The infinitesimal transformation for u_I is given by

$$\delta u_I = (D_I Q)\epsilon$$

where D_I is the total derivative operator

$$D_I = \frac{\partial}{x^I} + u_I \frac{\partial}{\partial u} + \sum_J u_{J,I} \frac{\partial}{\partial u_J} \qquad |J| > 0$$

with

$$\frac{\partial}{\partial x^I} = \frac{\partial^{|I|}}{\partial x_1^{i_1} \partial x_2^{i_2} \ldots \partial x_n^{i_n}}.$$

A.4. Invariant functions

A function $u(x)$ is said to be invariant if it is left unchanged by the action of the transformation group, that is $\partial u^\epsilon/\partial\epsilon = 0$, or equivalently

$$Q(x, u, \{u_i\}) = 0. \qquad \text{(A5)}$$

A.5. Symmetry group

A system of partial differential equations,

$$F(x, u, \{u_J\}) = 0 \qquad \text{(A6)}$$

is said to be invariant under a transformation group if every solution u is transformed by the group into other solution u^ϵ, that is $F(x, u^\epsilon, \{u_I^\epsilon\}) = 0$. The corresponding infinitesimal condition is

$$Q \frac{\partial F}{\partial u} + D_I(Q) \frac{\partial F}{\partial u_I} = 0 \qquad |I| > 0 \qquad \text{(A7)}$$

whenever u is a solution of the SPDE.

A.6. Invariant solutions

Invariant solutions are solutions of the SPDE that are invariant with respect to a symmetry group. Then they must be solutions of equations (A5) and (A6). When the SPDE models a physical system, invariant solutions are very often functions that exhibit interesting patterns with physical interest.

A.7. Procedure

In order to find a symmetry group of a SPDE we first substitute the partial differential equations into (A7). The resulting equations are treated as forms in the derivatives of u, whose coefficients depend on (u, x, t) and the infinitesimals (η, ξ). After the substitution we collect together the coefficients of like derivative terms in u and set all of them equal to zero. The resulting equations are called the determining equations of the group. In practice these equations are solvable and thus the infinitesimals and characteristics of the group are determined. The subsequent study is clearly shown in this paper.

A.8. Mathematical packages

These calculations, although not difficult in themselves, are clearly complicated as the order of the SPDE and the number of equations increase, so a software package for symbolic mathematics becomes really useful. To our knowledge, the best package for these kinds of calculations is Macsyma. Programs written by the authors in Macsyma 4.0, running in a Convex, have been used to obtain the results shown in this paper.

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